4.5 Solving Polynomial Equations

Finding solutions and zeros.

Recall the *Zero-Product Property* to solve factorable quadratic equations.

Example 1: Solving a Polynomial Equation by Factoring

Solve $2x^3 - 12x^2 + 18x = 0$

Look at the solutions! What do you notice about the zeros?

Key Notes!!!

When a factor $x - k$ of $f(x)$ is raised to an odd power the graph CROSSES the x-axis at $x = k$

When a factor $x - k$ of $f(x)$ is raised to an even power the graph TOUCHES the x-axis at $x = k$

Example 2: Finding Zeros of a Polynomial Function

Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. **Then sketch a graph of the function.**
(Hint: think of the equation as $f(x) = -2x^2 + 16x - 32$)

Rational Root Theorem!

Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

Example 3: Using the rational root theorem

Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$

Try on your own:

Find all real solutions of $x^3 + x^2 - 14x - 24 = 0$

Example 4: Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$

Key Notes!!!

Core Concept

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

Example 5: Using zeros to write a polynomial function.

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$

Homework
3-17 odd, 25, 28, 29, 33, 35, 39, 42, 43, 46, 50

4.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** If a polynomial function f has integer coefficients, then every rational solution of $f(x) = 0$ has the form $\frac{p}{q}$, where p is a factor of the ____________ and q is a factor of the ____________.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Find the y-intercept of the graph of $y = x^3 - 2x^2 - x + 2$.	Find the x-intercepts of the graph of $y = x^3 - 2x^2 - x + 2$.
Find all the real solutions of $x^3 - 2x^2 - x + 2 = 0$.	Find the real zeros of $f(x) = x^3 - 2x^2 - x + 2$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, solve the equation. *(See Example 1.)*

3. $z^3 - z^2 - 12z = 0$
4. $a^3 - 4a^2 + 4a = 0$
5. $2x^4 - 4x^3 = -2x^2$
6. $v^3 - 2v^2 - 16v = -32$
7. $5w^3 = 50w$
8. $9m^5 = 27m^3$
9. $2c^4 - 6c^3 = 12c^2 - 36c$
10. $p^4 + 40 = 14p^2$
11. $12n^2 + 48n = -n^3 - 64$
12. $y^3 - 27 = 9y^2 - 27y$

In Exercises 13–20, find the zeros of the function. Then sketch a graph of the function. *(See Example 2.)*

13. $h(x) = x^4 + x^3 - 6x^2$
14. $f(x) = x^4 - 18x^2 + 81$
15. $p(x) = x^6 - 11x^5 + 30x^4$
16. $g(x) = -2x^5 + 2x^4 + 40x^3$
17. $g(x) = -4x^4 + 8x^3 + 60x^2$
18. $h(x) = -x^3 - 2x^2 + 15x$
19. $h(x) = -x^3 - x^2 + 9x + 9$
20. $p(x) = x^3 - 5x^2 - 4x + 20$

21. **USING EQUATIONS** According to the Rational Root Theorem, which is *not* a possible solution of the equation $2x^4 - 5x^3 + 10x^2 - 9 = 0$?

    Ⓐ -9 Ⓑ $-\frac{1}{2}$ Ⓒ $\frac{5}{2}$ Ⓓ 3

22. **USING EQUATIONS** According to the Rational Root Theorem, which is *not* a possible zero of the function $f(x) = 40x^5 - 42x^4 - 107x^3 + 107x^2 + 33x - 36$?

    Ⓐ $-\frac{2}{3}$ Ⓑ $-\frac{3}{8}$ Ⓒ $\frac{3}{4}$ Ⓓ $\frac{4}{5}$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in listing the possible rational zeros of the function.

23. ✗ $f(x) = x^3 + 5x^2 - 9x - 45$

    Possible rational zeros of f: $1, 3, 5, 9, 15, 45$

24. ✗ $f(x) = 3x^3 + 13x^2 - 41x + 8$

    Possible rational zeros of f: $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{3}{8}$

In Exercises 25–32, find all the real solutions of the equation. *(See Example 3.)*

25. $x^3 + x^2 - 17x + 15 = 0$
26. $x^3 - 2x^2 - 5x + 6 = 0$

27. $x^3 - 10x^2 + 19x + 30 = 0$

28. $x^3 + 4x^2 - 11x - 30 = 0$

29. $x^3 - 6x^2 - 7x + 60 = 0$

30. $x^3 - 16x^2 + 55x + 72 = 0$

31. $2x^3 - 3x^2 - 50x - 24 = 0$

32. $3x^3 + x^2 - 38x + 24 = 0$

In Exercises 33–38, find all the real zeros of the function. *(See Example 4.)*

33. $f(x) = x^3 - 2x^2 - 23x + 60$

34. $g(x) = x^3 - 28x - 48$

35. $h(x) = x^3 + 10x^2 + 31x + 30$

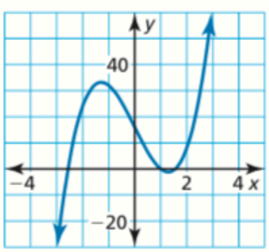
36. $f(x) = x^3 - 14x^2 + 55x - 42$

37. $p(x) = 2x^3 - x^2 - 27x + 36$

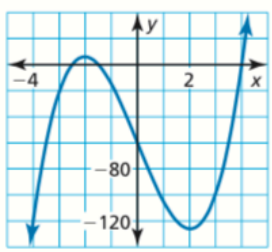
38. $g(x) = 3x^3 - 25x^2 + 58x - 40$

USING TOOLS In Exercises 39 and 40, use the graph to shorten the list of possible rational zeros of the function. Then find all real zeros of the function.

39. $f(x) = 4x^3 - 20x + 16$



40. $f(x) = 4x^3 - 49x - 60$



In Exercises 41–46, write a polynomial function *f* of least degree that has a leading coefficient of 1 and the given zeros. *(See Example 5.)*

41. $-2, 3, 6$

42. $-4, -2, 5$

43. $-2, 1 + \sqrt{7}$

44. $4, 6 - \sqrt{7}$

45. $-6, 0, 3 - \sqrt{5}$

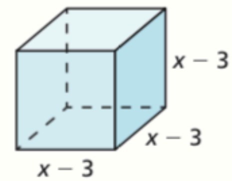
46. $0, 5, -5 + \sqrt{8}$

47. COMPARING METHODS Solve the equation $x^3 - 4x^2 - 9x + 36 = 0$ using two different methods. Which method do you prefer? Explain your reasoning.

48. REASONING Is it possible for a cubic function to have more than three real zeros? Explain.

49. PROBLEM SOLVING At a factory, molten glass is poured into molds to make paperweights. Each mold is a rectangular prism with a height 3 centimeters greater than the length of each side of its square base. Each mold holds 112 cubic centimeters of glass. What are the dimensions of the mold?

50. MATHEMATICAL CONNECTIONS The volume of the cube shown is 8 cubic centimeters.



a. Write a polynomial equation that you can use to find the value of x.

b. Identify the possible rational solutions of the equation in part (a).

c. Use synthetic division to find a rational solution of the equation. Show that no other real solutions exist.

d. What are the dimensions of the cube?

51. PROBLEM SOLVING Archaeologists discovered a huge hydraulic concrete block at the ruins of Caesarea with a volume of 945 cubic meters. The block is x meters high by $12x - 15$ meters long by $12x - 21$ meters wide. What are the dimensions of the block?

52. MAKING AN ARGUMENT Your friend claims that when a polynomial function has a leading coefficient of 1 and the coefficients are all integers, every possible rational zero is an integer. Is your friend correct? Explain your reasoning.

53. MODELING WITH MATHEMATICS During a 10-year period, the amount (in millions of dollars) of athletic equipment E sold domestically can be modeled by $E(t) = -20t^3 + 252t^2 - 280t + 21{,}614$, where t is in years.

a. Write a polynomial equation to find the year when about \$24,014,000,000 of athletic equipment is sold.

b. List the possible whole-number solutions of the equation in part (a). Consider the domain when making your list of possible solutions.

c. Use synthetic division to find when \$24,014,000,000 of athletic equipment is sold.