4.6 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

(First proven by German mathematician Carl Friedrich Gauss 1777-1855)

🅖 Core Concept

The Fundamental Theorem of Algebra

Theorem   If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary   If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

Example 1: Finding the number or solutions or zeros

1) How many solutions does the equation $x^3 + 3x^2 + 16x + 48 = 0$ have?

2) How many zeros does the function $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ have?

Example 2: Finding the zeros of a polynomial function

Find all zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$ (hint: use your calc to view the graph)

Complex Conjugates

Core Concept

The Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

Example 3: Using zeros to write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + i$

Rene' Descartes (1596-1650)

 Core Concept

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.

- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

Example 4: Using Descartes's rule of signs

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for:
$f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$

What are all the possible numbers of zeros for the function above:

Positive real zeros	Negative real zeros	Imaginary zeros	Total zeros
3	3	0	6
3	1	2	6
1	3	2	6
1	1	4	6

Example 5: Real World Applications

In Devin's mini cooper the tachometer measures the speed (in revolutions per minute, RPMs) at which an engine shaft rotates. For this certain car the speed x (in hundreds of RPMs) of the engine shaft and the speed s (in miles per hour) of the mini coop are modeled by:

$$s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11$$

What is the tachometer reading when the mini cooper travels 15mph?

(We all know Devin will never drive 15mph... 0 to 100... real quick!)

Homework
3-8, 10, 13, 15, 17-20, 22, 25, 28, 33, 36, 39, 44

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The expressions $5 + i$ and $5 - i$ are _____.

2. **WRITING** How many solutions does the polynomial equation $(x + 8)^3(x - 1) = 0$ have? Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, identify the number of solutions or zeros. *(See Example 1.)*

3. $x^4 + 2x^3 - 4x^2 + x = 0$ 4. $5y^3 - 3y^2 + 8y = 0$

5. $9t^6 - 14t^3 + 4t - 1 = 0$ 6. $f(z) = -7z^4 + z^2 - 25$

7. $g(s) = 4s^5 - s^3 + 2s^7 - 2$

8. $h(x) = 5x^4 + 7x^8 - x^{12}$

In Exercises 9–16, find all zeros of the polynomial function. *(See Example 2.)*
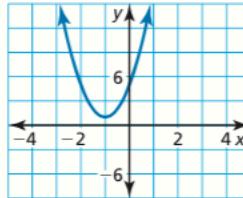
9. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$

10. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$

11. $g(x) = x^4 - 9x^2 - 4x + 12$

12. $h(x) = x^3 + 5x^2 - 4x - 20$

13. $g(x) = x^4 + 4x^3 + 7x^2 + 16x + 12$

14. $h(x) = x^4 - x^3 + 7x^2 - 9x - 18$

15. $g(x) = x^5 + 3x^4 - 4x^3 - 2x^2 - 12x - 16$

16. $f(x) = x^5 - 20x^3 + 20x^2 - 21x + 20$

ANALYZING RELATIONSHIPS In Exercises 17–20, determine the number of imaginary zeros for the function with the given degree and graph. Explain your reasoning.
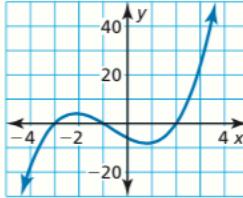
17. Degree: 4

18. Degree: 5
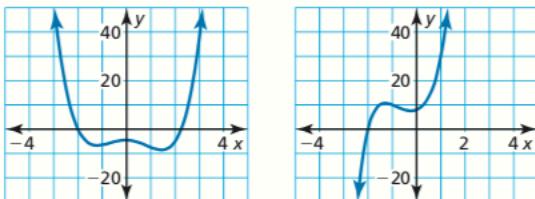
19. Degree: 2

20. Degree: 3

In Exercises 21–28, write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros. *(See Example 3.)*

21. $-5, -1, 2$

22. $-2, 1, 3$

23. $3, 4 + i$

24. $2, 5 - i$

25. $4, -\sqrt{5}$

26. $3i, 2 - i$

27. $2, 1 + i, 2 - \sqrt{3}$

28. $3, 4 + 2i, 1 + \sqrt{7}$

ERROR ANALYSIS In Exercises 29 and 30, describe and correct the error in writing a polynomial function with rational coefficients and the given zero(s).

29. Zeros: $2, 1 + i$

$$f(x) = (x - 2)[x - (1 + i)]$$
$$= x(x - 1 - i) - 2(x - 1 - i)$$
$$= x^2 - x - ix - 2x + 2 + 2i$$
$$= x^2 - (3 + i)x + (2 + 2i)$$

30. Zero: $2 + i$

$$f(x) = [x - (2 + i)][x + (2 + i)]$$
$$= (x - 2 - i)(x + 2 + i)$$
$$= x^2 + 2x + ix - 2x - 4 - 2i - ix - 2i - i^2$$
$$= x^2 - 4i - 3$$

31. **OPEN-ENDED** Write a polynomial function of degree 6 with zeros 1, 2, and $-i$. Justify your answer.

32. **REASONING** Two zeros of $f(x) = x^3 - 6x^2 - 16x + 96$ are 4 and -4. Explain why the third zero must also be a real number.

In Exercises 33–40, determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function. *(See Example 4.)*

33. $g(x) = x^4 - x^2 - 6$

34. $g(x) = -x^3 + 5x^2 + 12$

35. $g(x) = x^3 - 4x^2 + 8x + 7$

36. $g(x) = x^5 - 2x^3 - x^2 + 6$

37. $g(x) = x^5 - 3x^3 + 8x - 10$

38. $g(x) = x^5 + 7x^4 - 4x^3 - 3x^2 + 9x - 15$

39. $g(x) = x^6 + x^5 - 3x^4 + x^3 + 5x^2 + 9x - 18$

40. $g(x) = x^7 + 4x^4 - 10x + 25$

41. **REASONING** Which is *not* a possible classification of zeros for $f(x) = x^5 - 4x^3 + 6x^2 + 2x - 6$? Explain.

   (A) three positive real zeros, two negative real zeros, and zero imaginary zeros

   (B) three positive real zeros, zero negative real zeros, and two imaginary zeros

   (C) one positive real zero, four negative real zeros, and zero imaginary zeros

   (D) one positive real zero, two negative real zeros, and two imaginary zeros

42. **USING STRUCTURE** Use Descartes's Rule of Signs to determine which function has at least 1 positive real zero.

   (A) $f(x) = x^4 + 2x^3 - 9x^2 - 2x - 8$

   (B) $f(x) = x^4 + 4x^3 + 8x^2 + 16x + 16$

   (C) $f(x) = -x^4 - 5x^2 - 4$

   (D) $f(x) = x^4 + 4x^3 + 7x^2 + 12x + 12$

43. **MODELING WITH MATHEMATICS** From 1890 to 2000, the American Indian, Eskimo, and Aleut population P (in thousands) can be modeled by the function $P = 0.004t^3 - 0.24t^2 + 4.9t + 243$, where t is the number of years since 1890. In which year did the population first reach 722,000? *(See Example 5.)*

44. **MODELING WITH MATHEMATICS** Over a period of 14 years, the number N of inland lakes infested with zebra mussels in a certain state can be modeled by

$$N = -0.0284t^4 + 0.5937t^3 - 2.464t^2 + 8.33t - 2.5$$

where t is time (in years). In which year did the number of infested inland lakes first reach 120?

45. **MODELING WITH MATHEMATICS** For the 12 years that a grocery store has been open, its annual revenue R (in millions of dollars) can be modeled by the function

$$R = 0.0001(-t^4 + 12t^3 - 77t^2 + 600t + 13{,}650)$$

where t is the number of years since the store opened. In which year(s) was the revenue $1.5 million?

46. **MAKING AN ARGUMENT** Your friend claims that $2 - i$ is a complex zero of the polynomial function $f(x) = x^3 - 2x^2 + 2x + 5i$, but that its conjugate is *not* a zero. You claim that both $2 - i$ and its conjugate *must* be zeros by the Complex Conjugates Theorem. Who is correct? Justify your answer.

47. **MATHEMATICAL CONNECTIONS** A solid monument with the dimensions shown is to be built using 1000 cubic feet of marble. What is the value of x?