

## 6.3 Logarithms and Logarithmic Functions

**Do Now:** Solve for  $x$

1)  $2^x = 8$

2)  $5^x = 25$

3)  $2^x = 6$

**Logarithms:**  $\log_b y = x$  and  $b^x = y$

### Core Concept

#### Definition of Logarithm with Base $b$

Let  $b$  and  $y$  be positive real numbers with  $b \neq 1$ . The **logarithm of  $y$  with base  $b$**  is denoted by  $\log_b y$  and is defined as

$$\log_b y = x \quad \text{if and only if} \quad b^x = y.$$

The expression  $\log_b y$  is read as “log base  $b$  of  $y$ .”

#### Example 1: Rewriting Logarithmic Equations

Rewrite each equation in exponential form

a.  $\log_2 16 = 4$

b.  $\log_4 1 = 0$

c.  $\log_{12} 12 = 1$

d.  $\log_{1/4} 4 = -1$

#### Example 2: Rewrite Exponential Equations

Rewrite each equation in logarithmic form.

a.  $5^2 = 25$

b.  $10^{-1} = 0.1$

c.  $8^{2/3} = 4$

d.  $6^{-3} = \frac{1}{216}$

#### Example 3: Evaluating Logarithmic Expressions

Evaluate each logarithm

a.  $\log_4 64$

b.  $\log_5 0.2$

c.  $\log_{1/5} 125$

d.  $\log_{36} 6$

**Common Logarithms:** a logarithm with base 10 ( $\log_{10}$  or just  $\log$ )

**Natural Logarithms:** a logarithm with base e. ( $\log_e$  or just  $\ln$ )

**Example 4:** Evaluating Common and Natural Logarithms

Evaluate (a)  $\log 8$  and (b)  $\ln 0.3$  using a calculator. Round your answer to three decimal places.

**Work with a partner:**

**Rewrite the equation in exponential form.**

1.  $\log_3 81 = 4$       2.  $\log_7 7 = 1$       3.  $\log_{14} 1 = 0$       4.  $\log_{1/2} 32 = -5$

**Rewrite the equation in logarithmic form.**

5.  $7^2 = 49$       6.  $50^0 = 1$       7.  $4^{-1} = \frac{1}{4}$       8.  $256^{1/8} = 2$

**Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places.**

9.  $\log_2 32$       10.  $\log_{27} 3$       11.  $\log 12$       12.  $\ln 0.75$

## Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function  $g(x) = \log_b x$  is the inverse of the exponential function  $f(x) = b^x$ . This means that

$$g(f(x)) = \log_b b^x = x \quad \text{and} \quad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions “undo” each other.

**Example 5:** Using Inverse Properties

Simplify (a)  $10^{\log 4}$  and (b)  $\log_5 25^x$

**Example 6:** Finding Inverse Functions

Find the inverse of each function.

a)  $f(x) = 6^x$

b)  $y = \ln(x + 3)$

Work with a partner:

**13.**  $8^{\log_8 x}$

**14.**  $\log_7 7^{-3x}$

**15.**  $\log_2 64^x$

**16.**  $e^{\ln 20}$

**17.** Find the inverse of  $y = 4^x$ .

**18.** Find the inverse of  $y = \ln(x - 5)$ .

Homework

5-23odd, 26, 34, 35-53odd, 67

## 6.3 Exercises

Dynamic Solutions available at [BigIdeasMath.com](http://BigIdeasMath.com)

### Vocabulary and Core Concept Check

- COMPLETE THE SENTENCE** A logarithm with base 10 is called a(n) \_\_\_\_\_ logarithm.
- COMPLETE THE SENTENCE** The expression  $\log_3 9$  is read as \_\_\_\_\_.
- WRITING** Describe the relationship between  $y = 7^x$  and  $y = \log_7 x$ .
- DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

What power of 4 gives you 16?

What is log base 4 of 16?

Evaluate  $4^2$ .

Evaluate  $\log_4 16$ .

### Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, rewrite the equation in exponential form. (See Example 1.)

- $\log_3 9 = 2$
- $\log_4 4 = 1$
- $\log_6 1 = 0$
- $\log_7 343 = 3$
- $\log_{1/2} 16 = -4$
- $\log_3 \frac{1}{3} = -1$

In Exercises 11–16, rewrite the equation in logarithmic form. (See Example 2.)

- $6^2 = 36$
- $12^0 = 1$
- $16^{-1} = \frac{1}{16}$
- $5^{-2} = \frac{1}{25}$
- $125^{2/3} = 25$
- $49^{1/2} = 7$

In Exercises 17–24, evaluate the logarithm. (See Example 3.)

- $\log_3 81$
- $\log_7 49$
- $\log_3 3$
- $\log_{1/2} 1$
- $\log_5 \frac{1}{625}$
- $\log_8 \frac{1}{512}$
- $\log_4 0.25$
- $\log_{10} 0.001$

- NUMBER SENSE** Order the logarithms from least value to greatest value.

$\log_5 23$

$\log_6 38$

$\log_7 8$

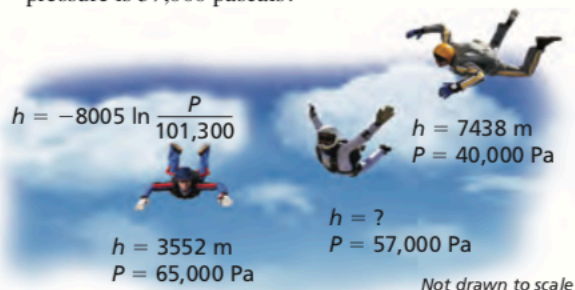
$\log_2 10$

- WRITING** Explain why the expressions  $\log_2(-1)$  and  $\log_1 1$  are not defined.

In Exercises 27–32, evaluate the logarithm using a calculator. Round your answer to three decimal places. (See Example 4.)

- $\log 6$
- $\ln 12$
- $\ln \frac{1}{3}$
- $\log \frac{2}{7}$
- $3 \ln 0.5$
- $\log 0.6 + 1$

- MODELING WITH MATHEMATICS** Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude  $h$  (in meters) above sea level is related to the air pressure  $P$  (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals?



- MODELING WITH MATHEMATICS** The pH value for a substance measures how acidic or alkaline the substance is. It is given by the formula  $\text{pH} = -\log[\text{H}^+]$ , where  $\text{H}^+$  is the hydrogen ion concentration (in moles per liter). Find the pH of each substance.

a. baking soda:  $[\text{H}^+] = 10^{-8}$  moles per liter

b. vinegar:  $[\text{H}^+] = 10^{-3}$  moles per liter

In Exercises 35–40, simplify the expression.  
(See Example 5.)

35.  $7^{\log_7 x}$

36.  $3^{\log_3 5x}$

37.  $e^{\ln 4}$

38.  $10^{\log 15}$

39.  $\log_3 3^{2x}$

40.  $\ln e^{x+1}$

41. **ERROR ANALYSIS** Describe and correct the error in rewriting  $4^{-3} = \frac{1}{64}$  in logarithmic form.

**X**  $\log_4(-3) = \frac{1}{64}$

42. **ERROR ANALYSIS** Describe and correct the error in simplifying the expression  $\log_4 64^x$ .

**X**  $\log_4 64^x = \log_4(16 \cdot 4^x)$   
 $= \log_4(4^2 \cdot 4^x)$   
 $= \log_4 4^{2+x}$   
 $= 2 + x$

In Exercises 43–52, find the inverse of the function.  
(See Example 6.)

43.  $y = 0.3^x$

44.  $y = 11^x$

45.  $y = \log_2 x$

46.  $y = \log_{1/5} x$

47.  $y = \ln(x - 1)$

48.  $y = \ln 2x$

49.  $y = e^{3x}$

50.  $y = e^{x-4}$

51.  $y = 5^x - 9$

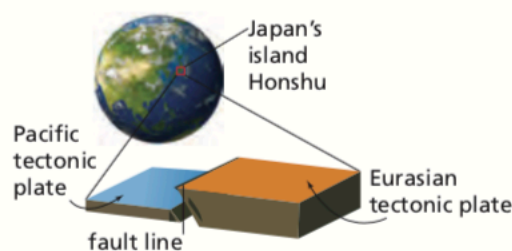
52.  $y = 13 + \log x$

53. **PROBLEM SOLVING** The wind speed  $s$  (in miles per hour) near the center of a tornado can be modeled by  $s = 93 \log d + 65$ , where  $d$  is the distance (in miles) that the tornado travels.

- a. In 1925, a tornado traveled 220 miles through three states. Estimate the wind speed near the center of the tornado.  
 b. Find the inverse of the given function. Describe what the inverse represents.



54. **MODELING WITH MATHEMATICS** The energy magnitude  $M$  of an earthquake can be modeled by  $M = \frac{2}{3} \log E - 9.9$ , where  $E$  is the amount of energy released (in ergs).



- a. In 2011, a powerful earthquake in Japan, caused by the slippage of two tectonic plates along a fault, released  $2.24 \times 10^{28}$  ergs. What was the energy magnitude of the earthquake?  
 b. Find the inverse of the given function. Describe what the inverse represents.

In Exercises 55–60, graph the function. (See Example 7.)

55.  $y = \log_4 x$

56.  $y = \log_6 x$

57.  $y = \log_{1/3} x$

58.  $y = \log_{1/4} x$

59.  $y = \log_2 x - 1$

60.  $y = \log_3(x + 2)$

**USING TOOLS** In Exercises 61–64, use a graphing calculator to graph the function. Determine the domain, range, and asymptote of the function.

61.  $y = \log(x + 2)$

62.  $y = -\ln x$

63.  $y = \ln(-x)$

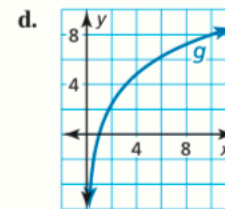
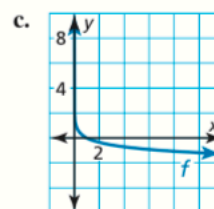
64.  $y = 3 - \log x$

65. **MAKING AN ARGUMENT** Your friend states that every logarithmic function will pass through the point  $(1, 0)$ . Is your friend correct? Explain your reasoning.

66. **ANALYZING RELATIONSHIPS** Rank the functions in order from the least average rate of change to the greatest average rate of change over the interval  $1 \leq x \leq 10$ .

a.  $y = \log_6 x$

b.  $y = \log_{3/5} x$

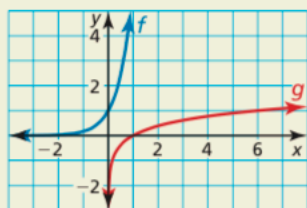


67. **PROBLEM SOLVING** Biologists have found that the length  $\ell$  (in inches) of an alligator and its weight  $w$  (in pounds) are related by the function  $\ell = 27.1 \ln w - 32.8$ .



- Use a graphing calculator to graph the function.
- Use your graph to estimate the weight of an alligator that is 10 feet long.
- Use the *zero* feature to find the  $x$ -intercept of the graph of the function. Does this  $x$ -value make sense in the context of the situation? Explain.

68. **HOW DO YOU SEE IT?** The figure shows the graphs of the two functions  $f$  and  $g$ .



- Compare the end behavior of the logarithmic function  $g$  to that of the exponential function  $f$ .
- Determine whether the functions are inverse functions. Explain.
- What is the base of each function? Explain.

69. **PROBLEM SOLVING** A study in Florida found that the number  $s$  of fish species in a pool or lake can be modeled by the function

$$s = 30.6 - 20.5 \log A + 3.8(\log A)^2$$

where  $A$  is the area (in square meters) of the pool or lake.



- Use a graphing calculator to graph the function on the domain  $200 \leq A \leq 35,000$ .
- Use your graph to estimate the number of species in a lake with an area of 30,000 square meters.
- Use your graph to estimate the area of a lake that contains six species of fish.
- Describe what happens to the number of fish species as the area of a pool or lake increases. Explain why your answer makes sense.

70. **THOUGHT PROVOKING** Write a logarithmic function that has an output of  $-4$ . Then sketch the graph of your function.

71. **CRITICAL THINKING** Evaluate each logarithm. (*Hint:* For each logarithm  $\log_b x$ , rewrite  $b$  and  $x$  as powers of the same base.)

- |                    |                 |
|--------------------|-----------------|
| a. $\log_{125} 25$ | b. $\log_8 32$  |
| c. $\log_{27} 81$  | d. $\log_4 128$ |

## Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Let  $f(x) = \sqrt[3]{x}$ . Write a rule for  $g$  that represents the indicated transformation of the graph of  $f$ . (Section 5.3)

72.  $g(x) = -f(x)$

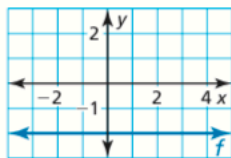
73.  $g(x) = f\left(\frac{1}{2}x\right)$

74.  $g(x) = f(-x) + 3$

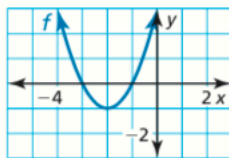
75.  $g(x) = f(x + 2)$

Identify the function family to which  $f$  belongs. Compare the graph of  $f$  to the graph of its parent function. (Section 1.1)

76.



77.



78.

