6.6 Solving Exponential and Logarithmic Equations

Exponential Equations: Equations in which the variable expression occurs as exponents

Example 1: Solving Exponential Equations
Solve each equation

a) $100^x = \left(\frac{1}{10}\right)^{x-3}$

b) $2^x = 7$

An important application of exponential equations is *Newton's Law of Cooling*. This law states that for a cooling substance with initial temperature T_0, the temperature T after t minutes can be modeled by

$$T = (T_0 - T_R)e^{-rt} + T_R$$

where T_R is the surrounding temperature and r is the cooling rate of the substance.

Example 2: Solving a Real-Life Problem
You are cooking a stew. When you take the stew off the stove its temperature is 212°F. The room temperature is 70°F, and the cooling rate of the stew is $r = 0.046$. How long will it take to cool the stew to a serving temperature of 100°F?

Logarithmic Equation: Are equations that involve logarithms of variable expressions.

Example 3: Solving Logarithmic Equations
Solve the following equations

a) $\ln(4x-7) = \ln(x+5)$

b) $\log_2(5x-17) = 3$

Example 4: Solving a Logarithmic Equation
Solve $\log 2x + \log(x-5) = 2$

Try on your own: Solve the equation. Check for extraneous solutions

5. $\ln(7x-4) = \ln(2x+11)$

6. $\log_2(x-6) = 5$

7. $\log 5x + \log(x-1) = 2$

8. $\log_4(x+12) + \log_4 x = 3$

Example 5: Solve an Exponential Inequality

Solve: $3^x < 20$

Example 6: Solving a Logarithmic Inequality

Solve: $\log x \leq 2$

Try on your own:

Solve the inequality.

9. $e^x < 2$ **10.** $10^{2x-6} > 3$ **11.** $\log x + 9 < 45$ **12.** $2 \ln x - 1 > 4$

Homework: 5, 7, 11, 14, 17, 18, 20, 21-31odd, 33, 34, 38, 47-53odd, 63,

6.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The equation $3^{x-1} = 34$ is an example of a(n) ___________ equation.

2. **WRITING** Compare the methods for solving exponential and logarithmic equations.

3. **WRITING** When do logarithmic equations have extraneous solutions?

4. **COMPLETE THE SENTENCE** If b is a positive real number other than 1, then $b^x = b^y$ if and only if _________.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–16, solve the equation. *(See Example 1.)*

5. $7^{3x+5} = 7^{1-x}$

6. $e^{2x} = e^{3x-1}$

7. $5^{x-3} = 25^{x-5}$

8. $6^{2x-6} = 36^{3x-5}$

9. $3^x = 7$

10. $5^x = 33$

11. $49^{5x+2} = \left(\frac{1}{7}\right)^{11-x}$

12. $512^{5x-1} = \left(\frac{1}{8}\right)^{-4-x}$

13. $7^{5x} = 12$

14. $11^{6x} = 38$

15. $3e^{4x} + 9 = 15$

16. $2e^{2x} - 7 = 5$

17. **MODELING WITH MATHEMATICS** The length ℓ (in centimeters) of a scalloped hammerhead shark can be modeled by the function

$$\ell = 266 - 219e^{-0.05t}$$

where t is the age (in years) of the shark. How old is a shark that is 175 centimeters long?

18. **MODELING WITH MATHEMATICS** One hundred grams of radium are stored in a container. The amount R (in grams) of radium present after t years can be modeled by $R = 100e^{-0.00043t}$. After how many years will only 5 grams of radium be present?

In Exercises 19 and 20, use Newton's Law of Cooling to solve the problem. *(See Example 2.)*

19. You are driving on a hot day when your car overheats and stops running. The car overheats at 280°F and can be driven again at 230°F. When it is 80°F outside, the cooling rate of the car is $r = 0.0058$. How long do you have to wait until you can continue driving?

20. You cook a turkey until the internal temperature reaches 180°F. The turkey is placed on the table until the internal temperature reaches 100°F and it can be carved. When the room temperature is 72°F, the cooling rate of the turkey is $r = 0.067$. How long do you have to wait until you can carve the turkey?

In Exercises 21–32, solve the equation. *(See Example 3.)*

21. $\ln(4x - 7) = \ln(x + 11)$

22. $\ln(2x - 4) = \ln(x + 6)$

23. $\log_2(3x - 4) = \log_2 5$

24. $\log(7x + 3) = \log 38$

25. $\log_2(4x + 8) = 5$

26. $\log_3(2x + 1) = 2$

27. $\log_7(4x + 9) = 2$

28. $\log_5(5x + 10) = 4$

29. $\log(12x - 9) = \log 3x$

30. $\log_6(5x + 9) = \log_6 6x$

31. $\log_2(x^2 - x - 6) = 2$

32. $\log_3(x^2 + 9x + 27) = 2$

In Exercises 33–40, solve the equation. Check for extraneous solutions. *(See Example 4.)*

33. $\log_2 x + \log_2(x - 2) = 3$

34. $\log_6 3x + \log_6(x - 1) = 3$

35. $\ln x + \ln(x + 3) = 4$

36. $\ln x + \ln(x - 2) = 5$

37. $\log_3 3x^2 + \log_3 3 = 2$

38. $\log_4(-x) + \log_4(x + 10) = 2$

39. $\log_3(x - 9) + \log_3(x - 3) = 2$

40. $\log_5(x + 4) + \log_5(x + 1) = 2$

ERROR ANALYSIS In Exercises 41 and 42, describe and correct the error in solving the equation.

41. ✗

$$\log_3(5x - 1) = 4$$
$$3^{\log_3(5x-1)} = 4^3$$
$$5x - 1 = 64$$
$$5x = 65$$
$$x = 13$$

42. ✗

$$\log_4(x + 12) + \log_4 x = 3$$
$$\log_4[(x + 12)(x)] = 3$$
$$4^{\log_4[(x+12)(x)]} = 4^3$$
$$(x + 12)(x) = 64$$
$$x^2 + 12x - 64 = 0$$
$$(x + 16)(x - 4) = 0$$
$$x = -16 \quad \text{or} \quad x = 4$$

43. PROBLEM SOLVING You deposit \$100 in an account that pays 6% annual interest. How long will it take for the balance to reach \$1000 for each frequency of compounding?

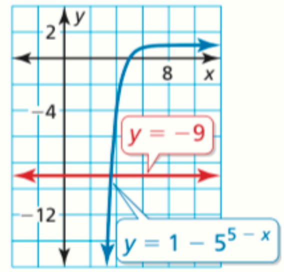
a. annual **b.** quarterly

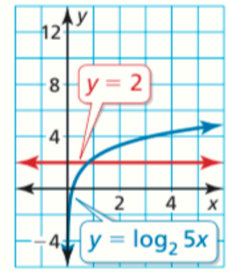
c. daily **d.** continuously

44. MODELING WITH MATHEMATICS The *apparent magnitude* of a star is a measure of the brightness of the star as it appears to observers on Earth. The apparent magnitude M of the dimmest star that can be seen with a telescope is $M = 5 \log D + 2$, where D is the diameter (in millimeters) of the telescope's objective lens. What is the diameter of the objective lens of a telescope that can reveal stars with a magnitude of 12?

45. ANALYZING RELATIONSHIPS Approximate the solution of each equation using the graph.

a. $1 - 5^{5-x} = -9$ **b.** $\log_2 5x = 2$





46. MAKING AN ARGUMENT Your friend states that a logarithmic equation cannot have a negative solution because logarithmic functions are not defined for negative numbers. Is your friend correct? Justify your answer.

In Exercises 47–54, solve the inequality. *(See Examples 5 and 6.)*

47. $9^x > 54$ **48.** $4^x \leq 36$

49. $\ln x \geq 3$ **50.** $\log_4 x < 4$

51. $3^{4x-5} < 8$ **52.** $e^{3x+4} > 11$

53. $-3 \log_5 x + 6 \leq 9$ **54.** $-4 \log_5 x - 5 \geq 3$

55. COMPARING METHODS Solve $\log_5 x < 2$ algebraically and graphically. Which method do you prefer? Explain your reasoning.

56. PROBLEM SOLVING You deposit \$1000 in an account that pays 3.5% annual interest compounded monthly. When is your balance at least \$1200? \$3500?

57. PROBLEM SOLVING An investment that earns a rate of return r doubles in value in t years, where $t = \dfrac{\ln 2}{\ln(1 + r)}$ and r is expressed as a decimal. What rates of return will double the value of an investment in less than 10 years?

58. PROBLEM SOLVING Your family purchases a new car for \$20,000. Its value decreases by 15% each year. During what interval does the car's value exceed \$10,000?

USING TOOLS In Exercises 59–62, use a graphing calculator to solve the equation.

59. $\ln 2x = 3^{-x+2}$ **60.** $\log x = 7^{-x}$

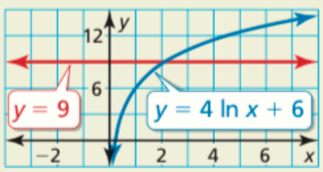
61. $\log x = 3^{x-3}$ **62.** $\ln 2x = e^{x-3}$

63. **REWRITING A FORMULA** A biologist can estimate the age of an African elephant by measuring the length of its footprint and using the equation $\ell = 45 - 25.7e^{-0.09a}$, where ℓ is the length (in centimeters) of the footprint and a is the age (in years).

a. Rewrite the equation, solving for a in terms of ℓ.

b. Use the equation in part (a) to find the ages of the elephants whose footprints are shown.

64. **HOW DO YOU SEE IT?** Use the graph to solve the inequality $4 \ln x + 6 > 9$. Explain your reasoning.



65. **OPEN-ENDED** Write an exponential equation that has a solution of $x = 4$. Then write a logarithmic equation that has a solution of $x = -3$.

66. **THOUGHT PROVOKING** Give examples of logarithmic or exponential equations that have one solution, two solutions, and no solutions.

CRITICAL THINKING **In Exercises 67–72, solve the equation.**

67. $2^{x+3} = 5^{3x-1}$

68. $10^{3x-8} = 2^{5-x}$

69. $\log_3(x - 6) = \log_9 2x$

70. $\log_4 x = \log_8 4x$

71. $2^{2x} - 12 \cdot 2^x + 32 = 0$

72. $5^{2x} + 20 \cdot 5^x - 125 = 0$

73. **WRITING** In Exercises 67–70, you solved exponential and logarithmic equations with different bases. Describe general methods for solving such equations.

74. **PROBLEM SOLVING** When X-rays of a fixed wavelength strike a material x centimeters thick, the intensity $I(x)$ of the X-rays transmitted through the material is given by $I(x) = I_0e^{-\mu x}$, where I_0 is the initial intensity and μ is a value that depends on the type of material and the wavelength of the X-rays. The table shows the values of μ for various materials and X-rays of medium wavelength.

Material	Aluminum	Copper	Lead
Value of μ	0.43	3.2	43

a. Find the thickness of aluminum shielding that reduces the intensity of X-rays to 30% of their initial intensity. (*Hint*: Find the value of x for which $I(x) = 0.3I_0$.)

b. Repeat part (a) for the copper shielding.

c. Repeat part (a) for the lead shielding.

d. Your dentist puts a lead apron on you before taking X-rays of your teeth to protect you from harmful radiation. Based on your results from parts (a)–(c), explain why lead is a better material to use than aluminum or copper.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write an equation in point-slope form of the line that passes through the given point and has the given slope. *(Skills Review Handbook)*

75. $(1, -2)$; $m = 4$

76. $(3, 2)$; $m = -2$

77. $(3, -8)$; $m = -\frac{1}{3}$

78. $(2, 5)$; $m = 2$

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. *(Section 4.9)*

79. $(-3, -50), (-2, -13), (-1, 0), (0, 1), (1, 2), (2, 15), (3, 52), (4, 125)$

80. $(-3, 139), (-2, 32), (-1, 1), (0, -2), (1, -1), (2, 4), (3, 37), (4, 146)$

81. $(-3, -327), (-2, -84), (-1, -17), (0, -6), (1, -3), (2, -32), (3, -189), (4, -642)$